JET PROPULSION LABORATORY **INTEROFFICE MEMORANDUM**

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TO: Distribution
FROM: Mark Milman
SUBJECT: Some preliminary astrometric modeling considerations

Introduction. This memo focuses on the astrometric reconstruction problem in "flatland", i.e., for stars and the SIM instrument constrained to a plane. In the model under consideration the celestial sphere is replaced by a circle, and a symmetric assumption is made on the viewing scenario. Specifically, it is assumed that each adjacent pair of stars on the circle is is observed with a fixed baseline of the instrument, and the normal to this baseline bisects the angle between the two stars. The resulting problem is simple enough to analytically characterize the solutions to the astrometry problem. These solutions when extrapolated to the three dimensional problem will hopefully lead to some useful insights regarding instrument field of view and calibration. (But perhaps your grandmother already knew these.)

Nevertheless, in summary form the main conclusions are: (i) When there is no scale error (the length of the baseline is known), when using a fixed baseline orientation it is advantageous to observe stars that are *close* together. For uniformly distributed stars on the unit circle with angle θ separating the nominal positions, the variance of the reconstruction error is proportional to $1/\cos^2(\theta/2)$. This is a manifestation of the instrument sensitivity and viewing scenario – to maintain a long projected baseline while observing two stars with the same baseline orientation requires the stars to be close to one another. (ii) For uniformly distributed stars the variance grows linearly with the number of stars. (iii) When the stars are uniformly distributed and there is an unknown scale error, the variance of the reconstructed positions is *unaffected* by the presence of the scale error. That is, the covariance matrix of the star positions and scale has the form $diag(\Omega, \alpha)$ where Ω is the star position covariance matrix and α is the variance of the scale error. The heuristic explanation for this result is that the perturbation in the measured delays introduced by a scale error cannot at all be compensated by perturbations in the star positions (when the nominal star positions are uniformly distributed). (iv) The variance of the scale error is proportional to $1/\sin^2(\theta/2)$ in the case of uniformly distributed stars, where θ again denotes the angle between adjacent star positions. (v) When the stars are not uniformly distributed, there is coupling between the scale error and the star position error. An exact expression for this variance is derived. The position error is shown to consist of a term that represents the error when there is no scale error plus another term contributed by a multiple of the scale error. The degree of this coupling error is shown to be directly related to the degree of nonuniformity of the star distribution. (vi) Given two observing scenarios, call them S_1 and S_2, if the separation angle between stars in S_1 is greater than in S_2, then the resulting baseline scale error for S_1 is less than the error for S_2. That is, the scale is more accurately estimated when stars are well separated.

These flatland conclusions are suggestive of a few conjectures. First the the only *intrinsic* advantage to increasing the size of the FOV is to reduce the scale error. There are, however, several derived advantages. A larger FOV allows more stars to be tied together with a single baseline measurement, thus permitting more observations per star. A second advantage is that grids that cover the celestial sphere can use fewer stars; hence mitigating the increased error introduced by having to estimate the positions of a greater number of stars. In the planar case we saw that the variance in the position grows linearly with the number of stars required to cover the circle. We conjecture that this growth for the 3–D problem is more benign. The basis for this conjecture comes from the observation that the normal equations that arise from the estimation problem look something like an approximate second difference operator (d^2/dx^2 in the linear case and the Laplacian, Δ, for a rectangular grid). The resulting variance is then related to the eigenvalues of these approximations. In one dimensional space the eigenvalues lead to a linear growth in the variance. If we could isometrically map the sphere onto the plane (which we cannot do), the distribution of the eigenvalues would lead to a logarithmic growth in the position variance. On the sphere we're not sure, but we guess that it would be sublinear as well.

This could possibly be exploited in choosing how to close the grid. Because closure with fewer stars leads to a smaller covariance error, it may be advantageous to incorporate "wide angle" loops consisting of few stars that form closed subgrids within the observation scenario. There is a trade–off between the sensitivity of the instrument (small angular separation between stars observed with the same baseline increases the sensitivity of the measurement), versus the number of stars required to get closure.

We conjecture that the relative decoupling between scale and position error which was exhibited in the planar case can similarly be achieved to some extent on the sphere. For example, the observation scenario that was defined in the plane has a 3–D analogue by pointing the instrument so that the normal to the baseline contains the barycenter of a triple of stars with the same baseline. (This particular model was suggested [1].)

For near term investigations several things come to mind in addition to checking out some of the conjectures described above. These include the interaction between the feedforward command and the notion of the regularized science baseline measurement, what calibration cycles need to be performed and when, and observability analysis.

The Setup. Let s_0, s_1, ..., s_N denote the directions of $N+1$ stars located on the unit circle. We assume the following observation sequence: For each successive pair of stars, s_i, s_{i+1}, there is an interferometer baseline vector b_i for which the pair of observations

$$y_{i1} = < s_i, b_i >, \tag{1a}$$

$$y_{i2} = < s_{i+1}, b_i >, \tag{1b}$$

is generated for $i = 1, ..., N$ with the periodic condition $s_{N+1} = s_1$. Because of *a priori* knowledge errors in the star directions and baseline vectors we seek corrections to s_i, b_j of the form

$$s_i \to s_i + \omega_i \times s_i, \quad \text{and} \quad b_j \to b_j + \omega^j \times b_j, \quad i = 1, ..., N, \tag{2}$$

where ω_i and ω^j are differential rotation vectors.

Using this linearization (1) becomes

$$y_{i1} = < s_i, b_i > + < s_i \times b_i, \omega_i > - < s_i \times b_i, \omega^i >, \tag{3a}$$

$$y_{i2} = < s_{i+1}, b_i > + < s_{i+1} \times b_i, \omega_{i+1} > - < s_{i+1} \times b_i, \omega^i > . \tag{3b}$$

In the sequel it will be assumed that for each pair of observations made with the baseline vector b_i the normal to b_i bisects the angle between the nominal star positions s_i and s_{i+1} so that $< s_i, b_i > = < s_{i+1}, b_i >$.

Uniform Star Distribution. Assume that the nominal positions of the stars are uniformly distributed over the unit circle. Thus there is an angle ψ such that

$$|s_i \times b_i| = |s_{i+1} \times b_i| = \cos \psi \tag{4}$$

for all i. (ψ is the angle between the normal to the baseline vector b_i and the star direction vector s_i.) This observation scenario is stringent, but not altogether ludricous. (For example, with a 10m baseline, if the *a priori* catalogue error is 5mas, a 40as pointing error leads to a 50pm delay error.) Equation (3) can then be assembled as the system

$$T \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega^N \end{pmatrix} = y, \quad \text{with} \quad y = \begin{pmatrix} y_{11} - < s_1, b_1 > \\ y_{12} - < s_2, b_1 > \\ \vdots \\ y_{N1} - < s_N, b_N > \\ y_{N2} - < s_1, b_N > \end{pmatrix}. \tag{5}$$

Adding a noise vector η to the observations results in the equation

$$y = Tx + \eta, \tag{6}$$

where x denotes the stacked vector consisting of the star position and baseline position perturbations. We will assume that $E(\eta\eta^T) = \sigma^2 I$, where $I = 2N \times 2N$ identity matrix.

Note that we can eliminate the unknowns ω^i by taking the difference of (3a) and (3b). Let C denote the matrix that does this operation for each pair of equations. Then $CT = |b|\cos\psi L$, where

$$L = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & & \cdots & -1 & 1 \end{pmatrix} \tag{7}$$

An easy calculation shows that $CC^T = 2I$, where I now denotes the $N \times N$ identity matrix. Thus after applying C, (6) becomes

$$Cy = |b|\cos\psi L\omega + \epsilon, \tag{8}$$

where $\omega = [\omega_1 \cdots \omega_N]^T$ and $\epsilon = C\eta$ with $E(\epsilon\epsilon^T) = 2I\sigma^2$. Let $w = [1 \cdots 1]^T$, and note that $Lw = 0$. In fact $N(L) = sp(w)$. Let U denote an $N \times (N-1)$ orthogonal matrix whose columns are all orthogonal to w. Since L has a nontrivial null space, we seek solutions to the least squares problem (or equivalently the minimum variance estimation problem)

$$\min_{\omega} E(||b|\cos\psi L\omega - Cy|^2) \quad \text{such that} \quad < \omega, w >= 0, \quad \omega = KCy, \tag{9}$$

for some matrix K. Although there are many ways to normalize ω to obtain a unique solution to (9), the condition $< \omega, w >= 0$ leads to the minimum variance solution over all normalizations of ω. With this normalization (8) becomes

$$Cy = |b|\cos\psi LUv + \epsilon, \tag{10}$$

and the minimum variance problem (9) becomes

$$\min_{v} E(||b|\cos\psi LUv - Cy|^2) \quad \text{such that} \quad v = KCy. \tag{11}$$

The solution to (9) is

$$\hat{\omega} = \frac{1}{|b|\cos\psi} U[U^T L^T LU]^{-1} U^T L^T Cy, \tag{12}$$

with covariance matrix

$$E((\omega - \hat{\omega})(\omega - \hat{\omega})^T) = \frac{2\sigma^2}{|b|^2 \cos^2 \psi} [U^T L^T L U]^{-1}. \tag{13}$$

We observe that $L^T L$ is the periodic tridiagonal matrix

$$L^T L = \begin{pmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & - & \cdots & 0 \\ \vdots & & & & \\ -1 & 0 & \cdots & -1 & 2 \end{pmatrix}, \tag{14}$$

for which the eigenvalues and eigenvectors are known (it is diagonalized by the Fourier matrix representing the discrete Fourier transform [2]). Hence, the optimal estimate and covariance in (12) and (13) can be computed in closed form. In particularn the variance of the estimate is derived from the sum of the reciprocals of the nonzero eigenvalues of $L^T L$:

$$E(|\omega - \hat{\omega}|^2) = \frac{2\sigma^2}{|b|^2 \cos^2 \psi} \sum_{j=1}^{N-1} \frac{1}{1 - \cos \frac{2\pi j}{N}}. \tag{15}$$

Two immediate observations from (15): (i) The variance is proportional to $1/|b|^2 \cos^2(\psi)$. (ii) From the approximation $1 - \cos h \approx h^2/2$, for large N the variance in position of each star grows linearly.

The Effect of Scale Error. Next we will assume that there is a small scale error in the baseline vector b_j. We replace the model (2) with

$$b_j \to b_j + \omega^j \times b_j + \delta b_j, \tag{16}$$

where δ is a fixed, small scalar. The linearization (3) with the included scale error has the form

$$y_{i1} =< s_i, b_i > + < s_i \times b_i, \omega_i > - < s_i \times b_i, \omega^i > + \delta < s_i, b_i >, \tag{17a}$$

$$y_{i2} =< s_{i+1}, b_i > + < s_{i+1} \times b_i, \omega_{i+1} > - < s_{i+1} \times b_i, \omega^i > + \delta < s_{i+1}, b_i > . \tag{17b}$$

Again we require the second order terms to have negligible contribution, e.g. $|\omega_i|\delta \approx 10^{-12}$m. (This bound is easily achieved since $|\omega_i| \approx 10^{-7}$, leads to $|\delta| \approx 10^{-5}$m.) Since $< s_i, b_i >= - < s_{i+1}, b_i >= |b| \sin \psi$, taking differences in (17) leads to

$$y_{i1} - y_{i2} - < s_i, b_i > + < s_{i+1}, b_i >= |b| \cos \psi (\omega_i - \omega_{i+1}) + 2|b| \sin \psi \delta. \qquad (18).$$

With the addition of the unknown scale error δ, the matrix L in (7) is augmented by the single column vector u, $u = 2|b| \sin \psi [1 \cdots 1]^T$ to have the form

$$\tilde{L} = [L \quad u]. \tag{19}$$

Since $Lu = 0$, $\text{rank}(\tilde{L})$ is one greater than $\text{rank}(L)$. (If not, then it is possible to solve the equation $Lx = u$; hence $u \in R(L) = N(L^T)^\perp$. But $N(L^T)$ is spanned by u, and the assertion holds.) The analogue to (11) is

$$Cy = [|b| \cos \psi L U \quad u] \begin{pmatrix} v \\ \delta \end{pmatrix} + \epsilon. \tag{20}$$

Let

$$M = [|b| \cos\psi LU \quad u]. \tag{21}$$

Then,

$$\begin{pmatrix} \hat{\omega} \\ \hat{\delta} \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} [M^T M]^{-1} M^T C y, \tag{22}$$

and

$$E((x-\hat{x})(x-\hat{x})^T)) = 2\sigma^2 [M^T M]^{-1}. \tag{23}$$

Now,

$$M^T M = \begin{pmatrix} |b|^2 \cos^2\psi U^T L^T L U & |b| cos\psi U^T L^T u \\ |b| \cos\psi u^T L U & |u|^2 \end{pmatrix}. \tag{24}$$

Since $L^T u = 0$, $M^T M$ is block diagonal, and its inverse is given simply by

$$[M^T M]^{-1} = \begin{pmatrix} \frac{1}{|b|^2 \cos^2\psi} [U^T L^T L U]^{-1} & 0 \\ 0 & \frac{1}{|u|^2} \end{pmatrix}. \tag{25}$$

Thus we see from (23) and (25) that the scale error and star position error are uncoupled. Note that the variance of the scale error is given by

$$\frac{2\sigma^2}{|u|^2} = \frac{\sigma^2}{2N|b|^2 \sin^2\psi},$$

which decreases linearly with the number of observations, but increases as the angle between adjacent star pairs decreases.

Another way of viewing the decoupling between the scale and position error is to realize that $N(L^T) = R(L)$, so that $< u, L\omega >= 0$ for all ω. What this says is that the contribution of the scale error to the measured delay (which is *constant* for all measurements) cannot be at all compensated by a perturbation of the estimated star positions.

The error between the positions and the scale couple as soon as $L^T u \neq 0$. This requires the stars to be in a nonuniform distribution. To model this distribution, let Θ represent the $N-$vector of angular separations between pairs of stars that are observed with the same baseline, $\Theta = [\theta_1 \cdots \theta_N]$, where θ_i = angle between s_i and s_{i+1}. Let $\psi_i = \theta_i/2$ and define D as the $N \times N$ diagonal matrix with $\cos\Psi = [\cos\psi_1 \cdots \cos\psi_N]$ on the diagonal. Also define the vector $v = 2|b|[\sin\psi_1 \cdots \sin\psi_N]^T$. The resulting observation equation is essentially as above in (20), but we replace L by DL, and u by the vector v. Hence, (24) becomes

$$M^T M = \begin{pmatrix} |b|^2 U^T L^T D^2 & |b| U^T L^T D v \\ |b| v^T D L U & |u|^2 \end{pmatrix}. \tag{26}$$

This time $M^T M$ is not block diagonal. In fact, $L^T D x = 0$ implies

$$x = [1/\cos\psi_1 \cdots 1/\cos\psi_N]^T, \tag{27}$$

and consequently $U^T L^T D v \neq 0$ and there will be coupling between the scale and position errors. This will now be characterized.

For notational convenience write $M^T M$ from (26) in the boardered matrix form

$$M^T M = \begin{pmatrix} \Sigma & g \\ g^T & \mu \end{pmatrix}. \tag{28}$$

Thus

$$[M^T M]^{-1} = \begin{pmatrix} A & w \\ w^T & \nu \end{pmatrix}, \quad \text{where} \tag{29}$$

$$A = [\Sigma - \frac{1}{\mu} g g^T]^{-1},$$

$$\nu = \frac{1}{\mu - < g, \Sigma^{-1} g >},$$

$$w = -\nu \Sigma^{-1} g.$$

The Sherman–Morrison [3] formula gives A as

$$A = \Sigma^{-1} + \nu \Sigma^{-1} g g^T \Sigma^{-1}. \tag{30}$$

The variance of the star positions is given by $tr(A)$, and the variance of the scale error is given by ν.

A somewhat more compact and revealing form for these can be obtained. Note that

$$\begin{aligned} < g, \Sigma^{-1} g > &= < U^T L^T D v, [U^T L^T D^2 L U]^{-1} U^T L^T D v > \\ &= < DLU(DLU)^+ v, v >, \end{aligned} \tag{31}$$

where $(DLU)^+$ is the pseudoinverse of DLU. But $DLU(DLU)^+$ is the projection onto the range space of DL. Denote this projection as $\Pi_{R(DL)}$. We thus obtain a simple characterization of ν (the variance of the scale error) as

$$\begin{aligned} \nu &= \frac{1}{|v|^2 - |\Pi_{R(DL)} v|^2} \\ &= \frac{1}{|\Pi_{N(L^T D^T)} v|^2}. \end{aligned} \tag{32}$$

Here $\Pi_{N(L^T D^T)}$ is the projection onto the null space of $L^T D$. (This follows from noting that $|v|^2 = |\Pi_{R(DL)} v|^2 + |\Pi_{N(L^T D^T} v|^2$.)

Next let $\hat{v}$ denote the least squares solution to

$$\min || b | DLUx - v|^2, \tag{33}$$

and observe that $\hat{v} = \Sigma^{-1} g$. Thus the variance of the star positions is given by $tr(A)$,

$$tr(A) = tr(\Sigma^{-1}) + \frac{|\hat{v}|^2}{|\Pi_{N(L^T D^T)} v|^2}. \tag{34}$$

The first term above is just the variance of the position error when there is no scale error. The second term shows the scale error contribution to the position error. This contribution depends fundamentally on $|\Pi_{N(L^T D^T)} v|^2$, and the error is minimized when this quantity is maximized. Recall that the null space of $L^T D$ is spanned by the vector x in (27). Thus,

$$|\Pi_{N(L^T D^T)} v|^2 = < \frac{x}{|x|}, v >^2 . \tag{35}$$

In the case of uniform star distribution, x is just a multiple of v and (35) above reduces to the previous result $|v|^2$. Now the components of x will typically not vary significantly from unity since because of FOV restrictions, $0 < \psi_i < \pi/8$ (this corresponds to a 45^o FOV), and $1/\cos\pi/8 \approx 1.08$. Hence, in general, the magnitude in (35) increases (and the scale error decreases) when the angular difference between the stars has a more uniform distribution.

Recognizing that the variance in the scale error is minimized when ν (hence (35)) is maximized, we will show that this error decreases as the separation angle increases. First we write

$$< \frac{x}{|x|}, v >^2 = (\sum \tan\psi_i)^2 \frac{1}{\sum \frac{1}{\cos^2\psi_i}}. \tag{36}$$

Suppose in two observing scenarios the angular separation between star pairs is given by the vectors Ψ^1 and Ψ^2 with $\Psi^1 < \Psi^2$ (where this inequality is understood to hold componentwise.) Let $v(\Psi) = [\sin\psi_1 \cdots \sin\psi_N]^T$, where $\Psi = [\psi_1 \cdots \psi_N]$. Define

$$F(\Psi) = < \frac{x}{|x|}, v(\Psi) >^2$$

We will show that $F(\Psi^1) < F(\Psi^2)$.

Let $z = \Psi^2 - \Psi^1$. For the moment, assume for each i that

$$\frac{\partial}{\partial\psi_i} < \frac{x}{|x|}, v >^2 = \frac{\partial F}{\partial\psi_i} \geq 0. \tag{37}$$

Define $F(\Psi(t)) = F(\Psi^1 + tz)$, and note that $F(\psi(0)) = F(\Psi^1)$ and $F(\Psi(1)) = F(\Psi^2)$. Furthermore,

$$F(\Psi^2) = F(\Psi^1) + \int_0^1 F'(\Psi(s))zds. \tag{39}$$

But

$$\begin{aligned} F'(\Psi(s))z &= \sum \frac{\partial F}{\partial\psi_i} z \\ &\geq 0 \end{aligned}, \tag{40}$$

since the components of z are all nonegative and we have assumed that $\partial F/\partial\psi_i$ is nonnegative. We will now show this to be the case by differentiating (36) with respect to ψ_i:

$$\frac{\partial}{\partial\psi_i}\{(\sum \tan\psi_j)^2 \frac{1}{\sum \frac{1}{\cos^2\psi_j}}\} = \frac{2\sum\tan\psi_j}{[\sum \frac{1}{\cos^2\psi_j}]^2 \frac{1}{\cos^2\psi_i}}\{\sum \frac{1}{\cos^2\psi_j} - \tan\psi_i \sum \tan\psi_j\} \tag{41}$$

For $\psi_j < \pi/2$ the terms outside the braces are positive. Also note for $\psi_i < \pi/4$ (which corresponds to a FOV less than $\pi/2$), $\tan\psi_i < 1$. And for each j,

$$\frac{1}{\cos^2\psi_j} - \tan\psi_j > 0.$$

Hence, the derivative is nonnegative, and indeed increasing the angular separation between stars reduces the variance of the scale error.

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Distribution

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